PHYSICAL REVIEW E VOLUME 55, NUMBER 4 APRIL 1997

Locally accurate dynamical Euclidean group

Étienne Forest
National Laboratory for High Energy Physics (KEK), 1-1 Oho, Tsukuba, Ibaraki 305, Japan
(Received 29 May 1996)

We derive the locally accurate representation for the dynamical symplectic group for a beam element immersed in a field-free region. The results are expressed in terms of the displacement of a fiducial frame in the usual Euclidean space. The method does not involve geometrical constructions of a complexity exceeding that of a usual change of basis in Euclidean space. The extra complexity is handled by algebraic manipulations connecting the Lie representation of the usual Euclidean group with its dynamical equivalent. This is achieved by eliminating potential divergences in the ''thin block'' representation. Although this representation is ideally suited for large machines, it fails in the neighborhood of 180° racetrack magnets due to these divergences. All operations described in this paper can be fully automatized in a computer code. [S1063-651X(97)11403-9]

PACS number(s): 29.20.Dh, 41.85.Ja, 41.85.Lc

I. INTRODUCTION

The introduction of the Euclidean group of translations and rotations in the realm of particle tracking codes has two purposes. First, in this day and age of ''object-oriented programming'' it forces an immediate crystallization of the concept of a map between two layout planes as the central object of a tracking code. A proper understanding of the theory will lead to a proper implementation on the computer. Second, it might also become important in small machines to know how to move magnets using techniques of greater generality. I should add that I have myself used approximate methods when appropriate, but find it satisfying to see them emerge from a correct theoretical framework.

In this paper we will try to establish a close connection between the image of a magnet as it might appear on a computer screen or in one's own brain and the actual map which propagates particles across the device. In particular, under certain conditions (magnet independence, Sec. II A), we will show a direct link between the rotational-translational properties of the image and that of the map.

The connection between the picture-object and map-object can be best understood by using simple analogies. For example, certain department stores have installed virtual reality systems to allow their rich clients to test the design of their future kitchen. The client, wearing a head-mounted display and special gloves, opens the doors of the various cabinets and drawers of the kitchen while an operator implements on the spot the client's suggestions. Clearly, in the case of a virtual reality kitchen, little of the kitchen functionality is programmed. While the client can open the door of the oven, he cannot cook a virtual turkey in it. At the other extreme, many of us have seen the ''holodeck'' of the popular science fiction series Star Trek: The Next Generation. In this virtual reality machine of the future, the computerized objects have not only shapes and forms, but have also the full functional attributes of their real counterpart; thus Captain Picard can really cook himself a virtual egg in his holodeck!

In the case of accelerator simulations, our theoretical goal is to set up a framework which is more than the department store setting, but certainly much less than that of the science fiction creation. Here, as in the case of the virtual kitchen, we will be able to grab a magnet and move it. However, as in the ''holodeck,'' we also derive the effect of the magnet displacements on the particle trajectory. In other words our framework, under the condition of magnet independence, will permit the realization of a virtual reality program in which an accelerator physicist grabs magnets, moves them, and watches the trajectory being drawn in front of his eyes, in 3D and in real time. Of course, that might require a lot of computer power, but above all it requires a clear understanding of the theory, so that the right computer classes can be written while we wait for the fast hardware.

By analogy to the kitchen, we need a virtual room in which to put our magnets and we need ideal fiducial frames on which the magnets are ideally located. This room is called the tunnel by accelerator physicists. We call the set of all fiducial frames the layout of the ring, just like the layout of the kitchen. This layout, as shown in Fig. 1, contains (Poincaré) surfaces of sections. These sections have frames of reference at locations O_1 and O_2. In a computer code these local frames can be described in terms of a universal frame at Ω; however, the tracking will be local and will not use Ω. In other words, the tracking code gives a prescription for carrying the state of the system (usually three coordinates and three momenta) from the frame at O_1 to the frame at O_2. This prescription, called a transfer map, we denote by $\vec{\zeta}_{12}$. Tracking proceeds iteratively in the obvious way: the results at O_2 are then propagated locally to the next surface of section at O_3 using $\vec{\zeta}_{23}$.

Now we imagine a physical object being lowered into the ring. In Fig. 2 we see a layout frame at O_{12} situated in the middle between the planes at O_1 and O_2. We will say that the magnet is in its ideal layout position if and only if the frame attached to the magnet at O' coincides with the frame of the layout at O_{12}. Moreover, for the purpose of tracking, we must attach to the magnet two frames, one at O'_1 and another one at O'_2, whose positions relative to O' are identical to the positions of O_1 and O_2 relative to O_{12}. We will assume that the person who wrote the tracking code did his job correctly: his routines give us the map $\vec{\zeta}_{1'2'}$ in the body frame attached to the magnet. It relates the coordinates at

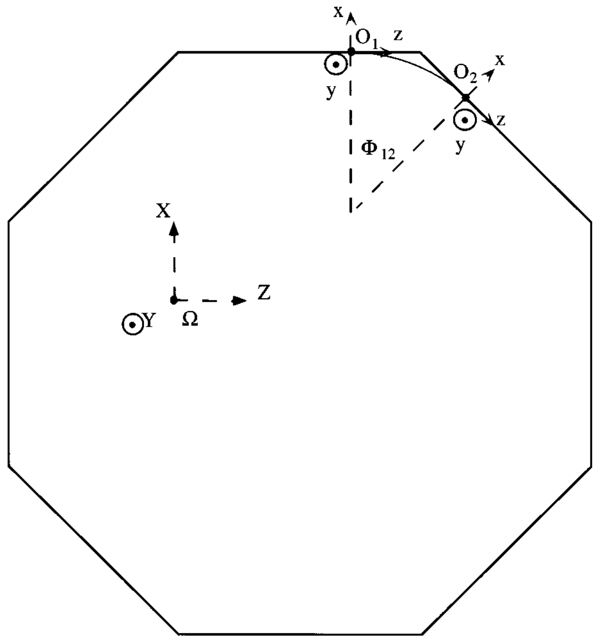

FIG. 1. Layout of a planar ring.

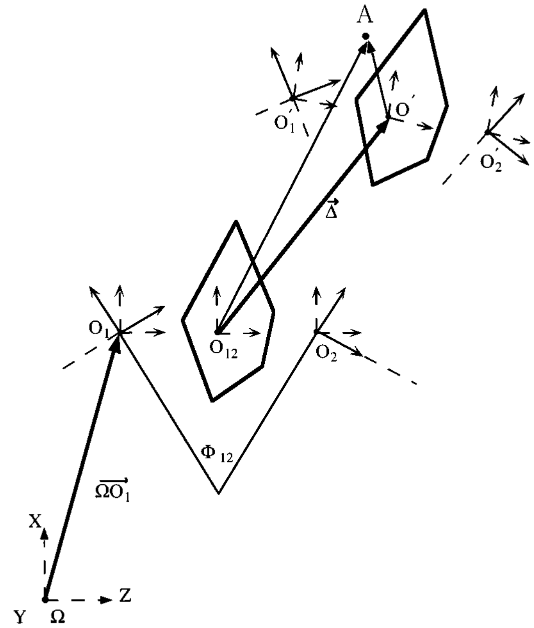

FIG. 2. Displacement of a magnet within the layout.

O'_2 to those at O'_1. For example, in the case of a normal quadrupole, the map $\vec{\zeta}_{1'2'}$ will have midplane symmetry. In the usual linear approximation, the transverse part of the map is made of two uncoupled two × two blocks. It really does not matter where this map will end up. The map $\vec{\zeta}_{1'2'}$ always remains the same. However, the actual map produced by this quadrupole in the actual ring may be different. For example, if a simple rotation of 90° around the longitudinal axis is performed, then the actual layout (or tunnel) map $\vec{\zeta}_{12}$ will be that of a skew quadrupole. In passing, we should say that the theory presented in this paper applies for any representation of the map: symplectic integration, Taylor series, etc.

The tracking code with no misalignments capability puts the prime body frames right on top of the layout frames. With our choice of frames and conventions, the ideal placement of the body map $\vec{\zeta}_{1'2'}$, which we denoted by I, is given by the formula

$$\vec{\zeta}_{12}=I[\vec{\zeta}_{1'2'}]=\vec{\zeta}_{1'2'}. \tag{1}$$

For our quadrupole example this means that a perfect normal quadrupole acts as one if placed in its ideal position in the tunnel. Notice that the placement map $I[\vec{\zeta}]$ is the identity. This is the result of a convention: the body frames and the layout frames are matched to one another.

The purpose of this paper is to derive the effect of a Euclidean transformation on the map $\vec{\zeta}_{1'2'}$, which is locally accurate. What does this mean? It is important to notice that the maps in a layout relate dynamical quantities not from a time t_1 to a second time t_2 but from a plane 1 to a second plane 2, which are approximately perpendicular to the actual physical tunnel. The existence of these maps assumes that the trajectories of interest move forward as time increases. This is not necessarily true for any possible trajectory. For example, a low energy particle entering a quadrupole off-axis could reverse direction if the field is sufficiently strong. The theory implemented in our code does not permit this. How is this restriction visible in an exact reformulation of the time-based dynamics (i.e., the usual equations of motion) into the layout dynamics (s-dependent Hamiltonian to use accelerator jargon)? It appears as a divergence in the various maps. Indeed, if we rotate a magnet in the plane of the ring, we should get into trouble as we approach 90° since the map attached to the magnet would then propagate rays in a direction perpendicular to the tunnel direction. Beyond 90° particles would actually reverse direction, which is absolutely forbidden. This implies that a rotation in the plane of the ring whose purpose is to rotate maps (not pictures) should contain a divergence at 90°. Thus the Euclidean group when applied to the dynamics of the layout (i.e., dynamical Euclidean group) cannot be a global representation of the group. Even though the usual group and its dynamical representation are isomorphic in the neighborhood of the origin in parameter space (Euler angles plus translations), the isomorphism cannot be global. Our discussion will start with the so-called ''thin block'' representation of the dynamical Euclidean group because its transformational properties are identical to that of a graphical object. Thus it is the simplest and most transparent representation of the dynamical Euclidean group. It is also sufficient in large machines because the angle Φ_{12} between layout planes is small.

Unfortunately, in the case of the thin map representation, unwanted singularities will appear even if the actual displacements of the magnet are infinitesimal. The source of these divergences is, as we will see, a rotation of magnitude equal to half the angle between the entrance and exit plane. This poses a serious problem for a bending angle near 180°, which can be found in small machines. We say that such a representation is not locally accurate. Our goal is to manipulate this thin representation so as to cancel the divergences: the representation is then locally accurate and expressed in terms of the usual graphical Euclidean group of 3×3 matrices and translations.

The rest of the paper is divided into four sections. The following section contains a discussion of the usual Euclidean group and its application to the usual time-based dynamics. In Sec. III, we discuss two possible ''transparent'' dynamical representations which are relatively easy to derive but not locally accurate (the thin block and the 180° representations). In Sec. IV we derive the locally accurate representation associated with our Euclidean representation. It is done with the help of the ''transparent thin block'' representation. Finally, in Sec. V, we reproduce some trivial results based on the small angle approximation using the locally accurate representation of Sec. IV.

II. ORDINARY EUCLIDEAN GROUP

A. Magnet independence

Equation (1) assumes that the map of a magnet does not depend on its location in a beam line. This is an idealization upon which most tracking codes rely. In reality, however, physically different magnets can interfere with one another if their fringe fields overlap. In that case they cease to be independent from a dynamical point of view. Indeed the map $\vec{\zeta}_{12}$ can depend on the presence of another magnet.

There is also the issue of space charge. One can show that it is not possible to define rigorously a propagator between layout planes for entities which are spatially extended and self-interacting. One can see this by imagining two strongly interacting particles: particle 1 is in the space between the layout planes at O_1 and O_2, while particle 2 has moved forward and is between the planes at O_2 and O_3. If the magnet between O_1 and O_2 is displaced, then propagation of the particle between O_2 and O_3 will be affected through the interparticle forces. This will be true even if particle 2 comes suddenly into existence in the O_2-O_3 magnet. It is most important to realize that particle 2 is not *directly* affected by the motion of the O_1-O_2 magnet since it is already in the O_2-O_3 magnet and, by assumption, it does not see the field from the O_1-O_2 magnet. However, the trajectory of particle 2 will be modified as it senses a different field coming from particle 1— different from the field it would have felt had the trajectory of particle 1 not been modified by the displacement of magnet O_1-O_2. Thus it is mathematically impossible to define an isolated propagator for each magnet.

We conclude that magnets can be interdependent in several ways: (i) mechanically by being physically linked to each other, (ii) magnetically by having their fields overlap, and (iii) dynamically by allowing strong interactions between particles.

Whatever source, this interdependence negates our ability to translate the operators of the Euclidean group acting on a graphical object (for example, the picture of our magnet in a CAD program) into a well-defined operation on the map $\vec{\zeta}_{12}$ propagating observables between the layout planes at O_1 and O_2. Thus from now on I will assume that the magnet of interest is independent. A body propagator attached to the prime frame can be defined, and for small motions of the prime frame within the layout, the body propagator is unaffected.

B. The geometrical object

Here I must first summarize results from Ref. [1] using a slightly different language. Originally the Euclidean group acts on points in our usual three-dimensional world. In the traditional geometry one can use coordinates in $\mathbb{R}^3$ to describe this space. These coordinates give us the location of a point with respect to a point O called the origin. Vectors can be defined vaguely as ''arrows'' originating at a point A and ending at a point B. This vector is called the vector $\overrightarrow{AB}$. Thus our vector space is isomorphic to $\mathbb{R}^3$ and its associated point (or affine) space requires $\mathbb{R}^3$ and an origin O. (Mathematicians define affine spaces out of vector spaces in a counterintuitive manner; here we intend to move ordinary graphical objects and thus our old Euclidean space of points is our fundamental space from which vectors emerge. We, human beings, live in an affine space of points, not in a vector space.)

A point A in an affine space can be located using three basis vectors in $\mathbb{R}^3$ and the origin O. To do this we consider the vector $\overrightarrow{OA}$, i.e., the ''arrow,'' between the origin O and point A. This vector can be written uniquely in a basis of the vector space $\mathbb{R}^3$. Thus we introduce a basis $(\vec{b}^1,\vec{b}^2,\vec{b}^3)$. There exists a unique set of coordinates $(\lambda_1,\lambda_2,\lambda_3)$ such that

$$\overrightarrow{OA}=\sum_{i=1}^{3}\lambda_i\vec{b}^i. \tag{2}$$

It should be noted that the origin can be chosen arbitrarily. The coordinates λ_i are called the coordinates of the point A in the affine basis $(O,\vec{b}^1,\vec{b}^2,\vec{b}^3)$.

The three-dimensional picture of a magnet, like any physical object, can be viewed as a set of points Q defined in a ''body frame'' attached to the magnet itself. In this paper we consider that the set of points Q belongs to a separate affine space, the space of the magnet. The action of placing a magnet in a ring consists in relating the points of the magnet to that of the layout through an isomorphism of affine space. Again it is useful to think in terms of a CAD program: each object allowed by the program (wood beam, steel beams, chairs, toilet bowls, etc.) must be defined internally independent of their final location. Typically the user moves this graphical object with a mouse or a virtual reality glove until it sits in its desired location. The same is true with our magnet. We must move it around until it sits between the appropriate layout planes. At this stage their is no dynamical meaning to all of this.

We say that a point $A\in Q$ has coordinates $(\lambda_1,\lambda_2,\lambda_3)$ in the body frame $(O',\vec{b}'^1,\vec{b}'^2,\vec{b}'^3)$ if and only if the vector $\overrightarrow{O'A}$ is given by

$$\overrightarrow{O'A}=\sum_{i=1}^{3}\lambda_i\vec{b}'^i. \tag{3}$$

The body frame is attached to the magnet, and moves with it. If, let us say, a dot is painted on the magnet, its coordinates in the body frame are a set of three numbers which will never change.

Mathematically the placement of a magnet in a tunnel can be viewed as an isomorphism between the affine space of the layout with its frame $(O_{12},\vec{b}^1_{12},\vec{b}^2_{12},\vec{b}^3_{12})$ and the affine space attached to the body of the magnet. In the case of a perfect placement, the affine basis $(O',\vec{b}'^1,\vec{b}'^2,\vec{b}'^3)$ is mapped into the layout basis $(O,\vec{b}^1_{12},\vec{b}^2_{12},\vec{b}^3_{12})$ which is located conveniently, but arbitrarily, between the two layout planes located around the magnet. These frames are depicted in Fig. 2. The point A on the magnet, let us say our painted dot, is mapped to a point B in the layout space (or tunnel) using the formula

$$\overrightarrow{OB}=I(\overrightarrow{O'A})=\sum_{i=1}^{3}\lambda_i I(\vec{b}'^i)$$
$$=\sum_{i,j=1}^{3}\lambda_i \vec{b}^i_{12}, \tag{4}$$

where the transformation I connects the two bases trivially. Equation (4) is the graphical equivalent of Eq. (1). I should emphasize again that the coordinates of A in the body frame are forever fixed, while its location in the tunnel (i.e., the point B) depends on our placement of the object. We are free to drop the magnet anywhere in the tunnel. Indeed, let us translate and rotate the magnet away from its ideal location by a transformation E. This is done by translating the origin of the magnet and rotating its axis vectors:

$$\overrightarrow{OB}=E(\overrightarrow{O'A})$$
$$=\vec{\Delta}+\sum_{i=1}^{3}\lambda_i R(\vec{b}'^i)$$
$$=\vec{\Delta}+\sum_{i,j=1}^{3}\lambda_i R_{ij}\vec{b}^j_{12}. \tag{5}$$

In Eq. (5), the origin of space is shifted by $\vec{\Delta}$, and the basis vectors attached to the body are mapped into a rotated set. The point B corresponds to the physical location in the tunnel of the point A attached to the magnet. Of course they are physically superimposed and one could purposely confuse them. Indeed one could view the entire process as a change of coordinates for a fixed point A in the layout: this is mathematically correct but does not fit well in our view of the magnet as an entity existing on its own. For example, the point A may refer to the position of our painted dot on the magnet while its position in the layout, B, may not remain constant as the magnet vibrates or is moved by humans.

Finally, let us express the effect of the Euclidean transformation E on the coordinates. In all the above formulas we have carefully avoided the usual confusion between a vector and its coordinates. For example, the vector $\vec{\Delta}$ refers to an actual displacement, not its coordinates. This displacement exists, is well defined, and is unique. It cannot depend on the choice of basis. This coordinate free language often seems very abstract to physicists: the language of differential geometry. However, it is the usual language of our everyday experience with the Euclidean group. Indeed when I tell my daughter ‘‘move your bike away from the door and put it against the fence,’’ the six year old understands immediately what translation should be performed. If I replace this statement by the phrase ‘‘move your bike in a direction perpendicular to the door frame by a distance of 3 meters,’’ it is unlikely that she will understand. I can change the basis to feet or centimeters, but it will still remain unclear. The vector $\vec{\Delta}$ belongs to this category of coordinate free statements in which the affine Euclidean group acts. However, being physicists, we will need a coordinate frame after all. Thus we express the final displaced coordinates in terms of the ideal coordinates. These ideal coordinates are just the body coordinates due to the convention of Eq. (4),

$$\overrightarrow{OB}=\sum_{i,j=1}^{3}\{\vec{\Delta}\cdot\vec{b}^j_{12}+\lambda_i R_{ij}\}\vec{b}^j_{12}$$
$$=\sum_{i,j=1}^{3}\{\vec{\Delta}\cdot\vec{b}^j_{12}+R^{-1}_{ji}\lambda_i\}\vec{b}^j_{12}$$
$$\Downarrow$$
$$E^{-1}(\vec{\lambda})=\vec{d}+R^{-1}\vec{\lambda}. \tag{6}$$

The ‘‘vectors’’ $\vec{d}$ and $\vec{\lambda}$ are coordinates (or components) describing the position of B in the layout basis at O_{12} (see again Fig. 2). For reasons that will soon be apparent, I choose to define the map E acting on the coordinates to be

$$E(\vec{\lambda})=R(\vec{\lambda}-\vec{d}). \tag{7}$$

C. Application to time-based dynamics in Cartesian coordinates

As we have said above, our propagators describe motion from one cross section of the tunnel to another. By contrast, ordinary Hamiltonian dynamics describes propagation from a time t_1 to a time t_2. If we apply a Euclidean transformation to a time-based propagator, then the entire ‘‘universe’’ is rotated or translated. Furthermore, since time is left invariant by this transformation, the actual transformation is a trivial extension of the usual geometrical picture. We simply have

$$E(\vec{q},\vec{p})=(R(\vec{q}-\vec{d}),R\vec{p}). \tag{8}$$

Now let us derive the effect of this Euclidean transformation by assuming that a single magnet composes our universe and that it is rotated according to Eq. (6). In the original configuration, before the application of a Euclidean transformation E, we have by assumption the propagator from time t_1 to time t_2:

$$\vec{z}_2=\vec{\zeta}_{t_1\to t_2}(\vec{z}_1),\quad \text{where } \vec{z}=(\vec{q},\vec{p}). \tag{9}$$

As we rotate the device, the map retains its functional form in coordinates $\vec{Z}$ attached to the magnet. Furthermore these coordinates will transform like the affine coordinates $\vec{\lambda}$ of the geometrical object. We thus have

$$E^{-1}(\vec{Z}_1)=\vec{z}_1$$

and

$$\vec{Z}_2 = \vec{\zeta}_{t_1\to t_2}(\vec{Z}_1)$$

$$\Downarrow$$

$$\vec{z}_2^{\,\rm new} = (E^{-1}\circ\vec{\zeta}_{t_1\to t_2}\circ E)(\vec{z}_1). \quad (10)$$

D. Connections to lie operators: Time-based dynamics

Our derivation of the locally correct representation of the Euclidean group will make use of an isomorphism between various Lie algebras. Thus let us make contact with Lie operators by introducing the new type of map $\mathcal{M}_{t_1\to t_2}$ associated with the transfer map $\vec{\zeta}_{t_1\to t_2}$. It is defined as follows: given an arbitrary function f of phase space,

$$\mathcal{M}_{t_1\to t_2} f = f\circ\vec{\zeta}_{t_1\to t_2}$$

or

$$\forall\vec{z}(\mathcal{M}_{t_1\to t_2} f)(\vec{z}) = f(\vec{\zeta}_{t_1\to t_2}(\vec{z})). \quad (11)$$

One can see in Eq. (11) that the map $\mathcal{M}_{t_1\to t_2}$ transforms an arbitrary function f of phase space into a new function $f\circ\vec{\zeta}_{t_1\to t_2}$. It does this by composing (i.e., substituting or ‘‘plugging in’’) the original function with the usual transfer map $\vec{\zeta}_{t_1\to t_2}$. It is this map $\mathcal{M}_{t_1\to t_2}$ which can be expressed in terms of Lie operators. For this reason we called such maps ‘‘compositional’’ maps. One should also point out that this definition as well as the existence of Lie operators extends to nonsymplectic maps as found in electron rings. In the case of symplectic compositional maps the Lie operators reduce to the usual Poisson bracket operators.

Defining a similar compositional map for the transformations E and E^{-1}, we conclude that the new compositional map $\mathcal{E}[\mathcal{M}_{t_1\to t_2}]$ associated with $E^{-1}\circ\vec{\zeta}_{t_1\to t_2}\circ E$ is given by

$$\mathcal{E}\,[\mathcal{M}_{t_1\to t_2}] = \mathcal{E}\mathcal{M}_{t_1\to t_2}\mathcal{E}^{-1}. \quad (12)$$

For example, if the map $\vec{\zeta}_{t_1\to t_2}$ is generated by a time-independent Hamiltonian H, then the solution for $\mathcal{M}_{t_1\to t_2}$ is just

$$\mathcal{M}_{t_1\to t_2} = \exp(:-(t_2-t_1)H:). \quad (13)$$

Here we follow the notation of Dragt for the Poisson bracket operator associated to a function f:

$$:f:g = [f,g]. \quad (14)$$

The transformed map $\mathcal{E}[\mathcal{M}_{t_0\to t_1}]$ is given by

$$\begin{aligned}\mathcal{E}\,[\mathcal{M}_{t_1\to t_2}] &= \mathcal{E}\mathcal{M}_{t_1\to t_2}\mathcal{E}^{-1}\\ &= \mathcal{E}\,\exp(:-(t_2-t_1)H:)\mathcal{E}^{-1}\\ &= \exp(\mathcal{E}:-(t_2-t_1)H:\mathcal{E}^{-1})\\ &= \exp(:-(t_2-t_1)H\circ E:). \quad (15)\end{aligned}$$

In the case of a time-dependent Hamiltonian, the results are identical since the Euclidean group does not affect time. Thus the rotated map is generated by the rotated Hamiltonian $H\circ E$:

$$(H\circ E)(\vec{q},\vec{p};t) = H(E(\vec{q},\vec{p};t)) = H(R(\vec{q}-\vec{d}),R\vec{p};t). \quad (16)$$

N.B. This definition does not assume symplecticity. If the map $\vec{\zeta}$ is generated by a force $\vec{F}$, it can still be rotated and the compositional map can still be defined using Eq. (11).

So far we have looked at the effect of the Euclidean group on a graphical representation of the magnet and on the time-based dynamics. In both cases the three rotations and three translations act linearly on either position space (graphical) or on the full phase space (in Cartesian coordinates). Our ultimate goal is to apply the Euclidean group on the map $\vec{\zeta}_{12}$ of the layout based dynamics.

III. EUCLIDEAN GROUP FOR THE LAYOUT: THIN AND RACETRACK REPRESENTATIONS

In the preceding section we introduced the action of the Euclidean group as it acts on the usual time-based Hamiltonian in Cartesian coordinates. Strictly speaking it is a ‘‘dynamical’’ representation: we have in Eq. (16) a prescription for rotating and translating the function H and thus the map it generates. However, the reader will agree that it is a trivial prescription. One needs only to double the dimensionality of the space since momenta will transform like positions under rotations. Translations are even simpler: they still only affect the positions.

The dynamical group associated with the layout dynamics is, on the other hand, less trivial. By introducing a map for each magnet, derived from a position-based Hamiltonian, we are now in a position to rotate and translate an individual magnet and its map. From the point of view of the global time-based H the magnet is a spatial fluctuation of the function H and the dynamical group of the layout rotates and translates these individual fluctuations. The condition of magnet independence discussed above insures that this local

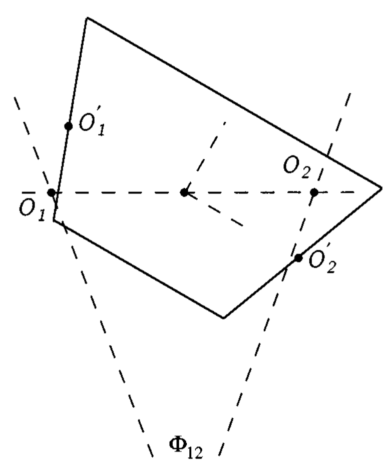

FIG. 3. Rotation of a bend and the failure of the usual formula of time-based dynamics. [See Eq. (12).]

fluctuation of the global H is rigidly attached to the physical magnet. Thus we should be able to derive the effect on the map in a way which parallels closely the effect on the graphical objects. In the jargon of object-oriented programming, we expect that it should be possible to use function overloading or virtual functions to relate the graphical and dynamical actions.

In this section I will summarize results obtained in Ref. [1] concerning the dynamical representation of the Euclidean group.

A. Failure of Eq. (12)

Consider a (bending) magnet placed in the layout between two planes bisecting each other at an angle Φ_{12}. In Fig. 3 we performed on it a simple rotation in the x-z plane. As one can see, it is incorrect to state that the rotated map $\mathcal{E}[\mathcal{M}_{12}]$ is given by (dropping the primes from now on)

$$\mathcal{E}[\mathcal{M}_{12}]=\mathcal{E}\mathcal{M}_{12}\mathcal{E}^{-1}. \tag{17}$$

[We assume that, prior to the application of $\mathcal{E}$, the map in its ideal position is matched to the layout as in Eq. (1).] The reasons for this are simple: after rotation, the entrance and exit frame of the body do not coincide with those of the layout. By analogy with time based dynamics, we say that an arbitrary Euclidean transformation will mix up the timelike coordinate of the layout representation (the usual ''s'' variable) with the dependent phase space variables.

B. The thin block method

In Ref. [1], we used a trick to rotate the magnet correctly. It can be seen that the failure of the usual formula is connected to the finite distance $O_1'O_2'$ between the planes attached to the body. Thus we should consider rotating a zero length magnet. In Ref. [1], we gave a formula for $\mathcal{E}[\mathcal{M}_{12}]$ derived using the thin magnet concept:

$$\mathcal{E}[\mathcal{M}_{12}]=\mathcal{Y}^{1/2}\mathcal{T}_z^{1/2}\mathcal{E}\underbrace{\mathcal{T}_z^{-1/2}\mathcal{Y}^{-1/2}\mathcal{M}_{12}\mathcal{Y}^{-1/2}\mathcal{T}_z^{-1/2}}_{\text{thin map}}\mathcal{E}^{-1}\mathcal{T}_z^{1/2}\mathcal{Y}^{1/2}. \tag{18}$$

The extraneous maps involved are a rotation in the x-z plane of half the layout angle and a drift of half the distance between the layout planes.

$$\mathcal{Y}^{\pm 1/2}=\exp\left(:\mp\frac{\Phi_{12}}{2}L_y:\right),$$

$$\mathcal{T}_z^{\pm 1/2}=\exp\left(:\pm\frac{O_1O_2}{2}p_z:\right). \tag{19}$$

In general, the operator $\mathcal{E}$ can be written as

$$\mathcal{E}_{\vec{d},\theta\vec{\beta}}=\mathcal{T}_{\vec{d}}\mathcal{R}_{\theta\vec{\beta}}=\exp(:\vec{d}\cdot\vec{p}:)\exp(\theta:\vec{\beta}\cdot\vec{L}:),$$

where

$$||\vec{\beta}||=1. \tag{20}$$

In summary, there are two differences between the time-based and layout-based representations: (i) the presence of operators to make the magnet ''thin,'' and (ii) the functional form of the generators $:\vec{p}:$ and $:\vec{L}:$. Indeed, the Euclidean transformations which affect the coordinate along the direction of propagation (''z'' or ''s'') are different and lead to nonlinearities as well as divergences for rotations of 90°. Here is a table of the various Lie generators.

	Time	Layout
p_z	p_z	$\sqrt{(1+\delta)^2-p_x^2-p_y^2}$
L_x	yp_z-zp_y	$y\sqrt{(1+\delta)^2-p_x^2-p_y^2}$
L_y	zp_x-xp_z	$-x\sqrt{(1+\delta)^2-p_x^2-p_y^2}$
L_z	xp_y-yp_x	xp_y-yp_x

(21)

C. Layout divergences: The need for a locally accurate representation

One can check that the Lie algebras of the time and the layout representation are identical. However, the transfer maps associated to these Lie transforms are quite different. Most important to our discussion is the map $\mathcal{Y}^{1/2}$. This compositional map is in fact a drift in polar coordinates; its associated transfer map is given by the formula

$$x^{\text{new}}=\frac{x}{\cos(\theta)\left(1-\frac{p_x\tan(\theta)}{p_z}\right)}, \tag{22}$$

$$p_x^{\text{new}}=p_x\cos(\theta)+\sin(\theta)p_z, \tag{23}$$

$$y^{\text{new}}=y+\frac{p_y x\tan(\theta)}{p_z\left(1-\frac{p_x\tan(\theta)}{p_z}\right)}, \tag{24}$$

$$p_y^{\text{new}}=p_y, \tag{25}$$

$$\ell^{\text{new}}=\ell+\frac{(1+\delta)x\tan(\theta)}{p_z\left(1-\frac{p_x\tan(\theta)}{p_z}\right)}, \tag{26}$$

where

$$p_z=\sqrt{(1+\delta)^2-p_x^2-p_y^2}.$$

We notice that this map becomes increasingly ill-behaved as we approach an angle θ of 90°. The problem is inherent to the thin map trick. In order to make the map $\mathcal{M}_{12}$ ''thin,'' one must change the layout planes by rotating them by half the layout angle. However, the layout angle is not completely arbitrary since it must be assumed that the trajectories of interest always go forward. We can see that application of the thin block method to an 180° racetrack magnet will always fail. One way to try to solve this problem is to derive, using geometry, formulas which apply to the 180° bend and use them for layout angles near 180°. It is possible to do such a thing because the geometry of a racetrack is simple. This result is given by the following complex set of formulas:

$$\mathcal{E}[\mathcal{M}_{12}]=\mathcal{Y}^{1/2}_{-\pi/2}\mathcal{T}_x^{-1/2}\mathcal{E}_{\pi/2}\underbrace{\mathcal{T}_x^{1/2}\mathcal{Y}^{-1/2}_{\pi/2}\mathcal{M}_{12}\mathcal{Y}^{-1/2}_{\pi/2}\mathcal{T}_x^{-1/2}}_{180^\circ\text{ racetrack}}\mathcal{E}^{-1}_{-\pi/2}\mathcal{T}_x^{1/2}\mathcal{Y}^{1/2}_{-\pi/2}, \tag{27}$$

where

$$\mathcal{Y}^{\pm 1/2}_{\mp\pi/2}=\exp\left(:\mp\frac{\pi-\Phi_{12}}{2}L_y:\right), \tag{28}$$

$$\mathcal{T}_x^{\pm 1/2}=\exp\left(:\pm\frac{O_1O_2}{2}p_x:\right), \tag{29}$$

$$\mathcal{E}_{\pi/2}=\exp(:d_xp_z+d_yp_y-d_zp_x:)\times\exp(\theta{:}\beta_xL_z+\beta_yL_y-\beta_zL_x:), \tag{30}$$

$$\mathcal{E}^{-1}_{-\pi/2}=\exp(\theta{:}\beta_xL_z-\beta_yL_y-\beta_zL_x:)\times\exp(:d_xp_z-d_yp_y-d_zp_x:). \tag{31}$$

One can see that the effect on an 180° layout can be gotten from the thin layout result by a mere relabeling scheme and a few minus signs. One can check that some obvious results are correctly predicted by the above formula. For example, if we apply it to a semicircular magnet, a displacement of d_x in the body prime frame corresponds to a drift of d_x at *both* the entrance and exit planes. By contrast, a straight element would be sandwiched between a trivial translation of d_x and its inverse as one would expect. The result for the semicircular racetrack is obvious to anyone who has played with a train set.

With the thin lens representation and the racetrack representation, we have somewhat reduced the gravity of the problem. Clearly, for maps with layout angle around 90°, our two representations will be equally good (or bad) on average. Certain trajectories will be lost in the manipulations even if the effect of the Euclidean transformation is near identity. Thus we are lead to the ''locally accurate representation.''

IV. THE LOCALLY ACCURATE REPRESENTATION

The techniques we will use involve the manipulation and metamorphosis of ill-defined operators into a well-defined one. First let us apply this technique to link the thin map and 180° representations.

A. Commuting ill-defined operators

As I have said, the racetrack representation can be derived by pure geometry and was indeed first derived by pure geometry. However, it is instructive to derive it from the thin map representation by assuming that the dynamical and geometrical Lie groups are isomorphic (which they are) and both global (the dynamical group is not). The idea is to commute the operator $\sigma=\exp(:-(\pi/2)L_y:)$ past the various operators of the thin map representation. This is a suspicious operation since the dynamical version of this operator is ill-defined. But, using the isomorphism, let us derive the effect of the operator σ using the time-based version of our group (which includes the geometrical group trivially). First we get the following trivial results for σ:

$$\begin{aligned} x &\overset{\sigma^{-1}}{\rightarrow} z,\\ p_x &\rightarrow p_z,\\ z &\rightarrow -x,\\ p_z &\rightarrow -p_x. \end{aligned} \tag{32}$$

Now let us take the operators on the left side of $\mathcal{M}_{12}$ in Eq. (18) and introduce the identity map as $\sigma\sigma^{-1}$:

$$\begin{aligned} &\mathcal{Y}^{1/2}\mathcal{T}_z^{1/2}\mathcal{E}\mathcal{T}_z^{-1/2}\mathcal{Y}^{-1/2}\\ &\quad=\mathcal{Y}^{1/2}\sigma\sigma^{-1}\mathcal{T}_z^{1/2}\sigma\sigma^{-1}\mathcal{E}\sigma\sigma^{-1}\mathcal{T}_z^{-1/2}\sigma\sigma^{-1}\mathcal{Y}^{-1/2}\\ &\quad=\mathcal{Y}^{1/2}_{-\pi/2}\exp\left(:\frac{O_1O_2}{2}\sigma^{-1}p_z:\right)\sigma^{-1}\mathcal{E}\sigma\\ &\qquad\times\exp\left(:-\frac{O_1O_2}{2}\sigma^{-1}p_z:\right)\mathcal{Y}^{-1/2}_{\pi/2}. \end{aligned} \tag{33}$$

The operator $\sigma^{-1}\mathcal{E}\sigma$ is also evaluated by moving σ into the exponent:

$$\begin{aligned} \sigma^{-1}\mathcal{E}\sigma&=\sigma^{-1}\exp(:\vec{d}\cdot\vec{p}:)\exp(\theta{:}\vec{\beta}\cdot\vec{L}:)\sigma\\ &=\exp(:\vec{d}\cdot\sigma^{-1}\vec{p}:)\exp(\theta{:}\vec{\beta}\cdot\sigma^{-1}\vec{L}:). \end{aligned} \tag{34}$$

We must evaluate the effect of σ on the six Lie generators of $\mathcal{E}$. Using Eq. (32) we conclude that

$$\begin{aligned} x &\overset{\sigma^{-1}}{\rightarrow} z, \\ p_x &\rightarrow p_z, \\ p_z &\rightarrow -p_x, \\ L_x &\rightarrow L_z, \\ L_z &\rightarrow -L_x. \end{aligned} \tag{35}$$

Thus Eq. (33) is further modified into

$$\begin{aligned} \mathcal{Y}^{1/2}\mathcal{T}_z^{1/2}\mathcal{E}\mathcal{T}_z^{-1/2} &= \mathcal{Y}_{-\pi/2}^{1/2}\exp\left(:\frac{O_1O_2}{2}\sigma^{-1}p_z:\right)\sigma^{-1}\mathcal{E}\sigma \\ &\quad\times\exp\left(:-\frac{O_1O_2}{2}\sigma^{-1}p_z:\right)\mathcal{Y}_{\pi/2}^{-1/2} \\ &= \mathcal{Y}_{-\pi/2}^{1/2}\exp\left(:-\frac{O_1O_2}{2}p_x:\right)\sigma^{-1}\mathcal{E}\sigma \\ &\quad\times\exp\left(:\frac{O_1O_2}{2}p_x:\right)\mathcal{Y}_{\pi/2}^{-1/2} \\ &= \mathcal{Y}_{-\pi/2}^{1/2}\mathcal{T}_x^{-1/2}\mathcal{E}_{\pi/2}\mathcal{T}_x^{1/2}\mathcal{Y}_{\pi/2}^{-1/2}. \end{aligned} \tag{36}$$

The right-hand side of Eq. (18) can also be manipulated so as to produce the rest of the 180° formula.

B. Derivation of the locally accurate representation

Now one must ask how the miracle happened in the preceding section. Strictly speaking, the isomorphism does not apply if the operators multiply each other into one of the rotations, which is not defined in the dynamical representation. However, they apply to operators in the neighborhood of ill-defined operators even though the validity of the dynamical representation is restricted to an area of phase space which is vanishingly small as we approach critical anglest such as 90°. The idea behind the ''locally accurate representation'' is to remove from the thin map representation all the maps which are ''large,'' i.e., those that depend on the layout angle. This is done by moving them into the exponents and then using the isomorphism to recompute the exponents.

Proceeding as if we are in the time based representation, we derive a trivial identity concerning the commutation of translations and rotations (summing over repeated indices):

$$\begin{aligned} \mathcal{R}_{\theta\vec{\beta}}\mathcal{T}_{\vec{d}}q_i &= \mathcal{R}_{\theta\vec{\beta}}\{q_i-d_i\} \\ &= \{R_{ij}q_j-d_i\} \\ &= R_{ij}\{q_j-R_{jk}^{-1}d_k\} \\ &= \mathcal{T}_{R^{-1}\vec{d}}\{R_{ij}q_j\} \\ &= \mathcal{T}_{R^{-1}\vec{d}}\mathcal{R}_{\theta\vec{\beta}}q_i. \end{aligned} \tag{37}$$

We first apply this formula to the left-hand side of Eq. (18),

$$\mathcal{T}_z^{1/2}\mathcal{E}\mathcal{T}_z^{-1/2}=\mathcal{T}_{\vec{D}}\mathcal{E},$$

where

$$\vec{D}=(1-R_{\theta\vec{\beta}}^{-1})\begin{pmatrix}0\\0\\O_{12}/2\end{pmatrix}, \tag{38}$$

and then to the right-hand side

$$\mathcal{T}_z^{-1/2}\mathcal{E}^{-1}\mathcal{T}_z^{1/2}=\mathcal{E}^{-1}\mathcal{T}_{\vec{D}}. \tag{39}$$

Now we substitute Eqs. (38) and (39) into Eq. (18):

$$\mathcal{E}[\mathcal{M}_{12}]=\mathcal{Y}^{1/2}\mathcal{T}_{\vec{D}}\mathcal{E}\mathcal{Y}^{-1/2}\mathcal{M}_{12}\mathcal{Y}^{-1/2}\mathcal{T}_z^{-1/2}\mathcal{E}^{-1}\mathcal{T}_{\vec{D}}\mathcal{Y}^{1/2}. \tag{40}$$

To proceed further we use the following property of rotations: If

$$\underbrace{\exp\left(\gamma:\vec{\alpha}\cdot\vec{L}:\right)}_{\mathcal{R}_{\gamma\vec{\alpha}}}\vec{q}=R_{\gamma\vec{\alpha}}\vec{q},$$

then

$$\mathcal{R}_{\gamma\vec{\alpha}}\exp(\theta:\vec{\beta}\cdot\vec{L}:)\mathcal{R}_{\gamma\vec{\alpha}}^{-1}=\exp(\theta:\{R_{\gamma\vec{\alpha}}^{-1}\vec{\beta}\}\cdot\vec{L}:). \tag{41}$$

This allows us to get rid of the map $\mathcal{Y}^{1/2}$ by moving it into the exponent of $\mathcal{E}$:

$$\begin{aligned} &\mathcal{E}_{\vec{d},\theta\vec{\beta}}[\mathcal{M}_{12}] \\ &\quad=\mathcal{Y}^{1/2}\mathcal{T}_{\vec{D}+\vec{d}}\mathcal{Y}^{-1/2}\mathcal{R}_{\theta Y^{-1/2}\vec{\beta}}\mathcal{M}_{12}\mathcal{R}_{-\theta Y^{1/2}\vec{\beta}}\mathcal{Y}^{-1/2}\mathcal{T}_{\vec{D}-\vec{d}}\mathcal{Y}^{1/2} \\ &\quad=\mathcal{T}_{Y^{-1/2}\{\vec{D}+\vec{d}\}}\mathcal{R}_{\theta Y^{-1/2}\vec{\beta}}\mathcal{M}_{12}\mathcal{R}_{-\theta Y^{1/2}\vec{\beta}}\mathcal{T}_{Y^{1/2}\{\vec{D}-\vec{d}\}} \\ &\quad=\mathcal{T}_{\vec{\Delta}^{\rm in}}\mathcal{R}_{\theta\vec{\beta}^{\rm in}}\mathcal{M}_{12}\mathcal{R}_{-\theta\vec{\beta}^{\rm out}}\mathcal{T}_{-\vec{\Delta}^{\rm out}} \\ &\quad=\mathcal{E}_{\vec{\Delta}^{\rm in},\theta\vec{\beta}^{\rm in}}\mathcal{M}_{12}\mathcal{E}^{-1}_{\vec{\Delta}^{\rm out},\theta\vec{\beta}^{\rm out}}. \end{aligned} \tag{42}$$

These new inputs are all computable using simple linear algebra,

$$\vec{\Delta}^{\rm in}=Y^{-1/2}\{\vec{D}+\vec{d}\}, \tag{43}$$

$$\vec{\beta}^{\rm in}=Y^{-1/2}\vec{\beta}, \tag{44}$$

$$\vec{\Delta}^{\rm out}=Y^{1/2}\{\vec{D}-\vec{d}\}, \tag{45}$$

$$\vec{\beta}^{\rm out}=Y^{1/2}\vec{\beta}, \tag{46}$$

$$\vec{D}=(1-R_{\theta\vec{\beta}}^{-1})\begin{pmatrix}0\\0\\O_{12}/2\end{pmatrix}. \tag{47}$$

$Y^{1/2}$ is a 3×3 matrix corresponding to a rotation of Φ_{12} in the x-z plane. $Y^{-1/2}$ is its inverse.

C. What have we achieved?

The final form for the displaced map $\mathcal{E}_{\vec{d},\theta\vec{\beta}}[\mathcal{M}_{12}]$ consists of a dynamical operator acting at the entrance and a dynamical operator acting at the exit. These operators have the most obvious physical interpretation. The first one, $\mathcal{E}_{\vec{\Delta}^{\rm in},\theta\vec{\beta}^{\rm in}}$, is a collection of drifts and coordinate changes necessary to move a particle from the layout frame at O_1 to the body frame at O_1'. The second operator, $\mathcal{E}^{-1}_{\vec{\Delta}^{\rm out},\theta\vec{\beta}^{\rm out}}$, is a collection of drifts and coordinate changes necessary to move a particle from the body frame at O_2' to the layout frame at O_2. All the factors entering in the construction of these operators are small if the displacement of the magnet is small. Therefore these maps are locally accurate to the extent that the amount of phase space lost due to the nonlocality of the dynamical representation is proportional to the actual displacement and not to the layout angle Φ_{12}.

Moreover we have derived the local dynamical results directly from the geometrical operator. The geometrical operator requires one fiducial frame of reference attached to the magnet which, in the ideal positioning, is superimposed on a fiducial frame of the layout. The displacements of the geometrical object are an exercise in relating these two frames. By using the thin magnet trick, we can use the geometrical formulas to produce the dynamical representation. Then, by using our isomorphisms, we commute all ''big operators'' necessary for the thin map trick until they cancel one another. We have then succeeded in relating layout planes to body planes in terms of the geometrical fiducial frames without using mind boggling geometrical constructions. Our methods have two clear advantages, as follows.

(i) If someone decides to change the fiducial frames, the results of this paper provide an immediate algebraic prescription for the derivation of the locally accurate representation in terms of the new frames. One first produces the thin map representation in this new frame and second one commutes the layout operators using the above techniques. At no point does the geometry needed for the dynamical representation exceed in complexity that of the geometrical objects.

(ii) In addition, the spirit behind our methods fits the new ''object oriented'' programming methods very well. It elevates the map of the magnet and the geometrical picture to the status of ''objects'' within the new programming paradigm. Obviously if a computer code knows how to picture the magnet on a screen as it moves it (CAD procedure), it also knows (through function overloading, for example) how to propagate in the layout across the displaced element.

On the last point it important to remember that the dynamical structure of the map was of paramount importance in the elevation of the map to the status of an independent object. A magnet can propagate self-interacting particles and still remain a perfectly well-defined geometrical object, but ceases to be a self-contained dynamical object. Of course if the self-interacting forces are sufficiently localized (beam-beam for example), one can cheat and retain the dynamical object in first approximation.

V. TRIVIAL APPLICATION: THE TRANSVERSE TILT IN LARGE MACHINES

Many tracking codes in accelerator physics assume that the small angle approximation holds. This is so ingrained in the culture that it has become customary to refer to bends and quadrupoles as ''linear elements.'' In addition, the angle Φ_{12} of the layout is also assumed to be small. Under these conditions it is interesting to derive an approximate formula for the transverse tilt of an element situated between layout planes for which $\Phi_{12}\neq 0$.

Let us start with the ''back-of-the-envelope'' derivation of this effect. First it is assumed that if Φ_{12} is not zero, then there must exist in this layout slot a bending element which bends the ideal particle by Φ_{12}. Let us call the map for this element $\mathcal{S}$. In units of magnetic rigidity (B_ρ), the value of the B field is just $1/\rho_{12}$, the curvature of an ideal trajectory between plane 1 and plane 2. Then it is postulated that the rotation of this element along the z axis by an angle θ_z has two components: first there is a change of coordinates at the entrance and exit (as in a straight element) given by a rotation $\mathcal{R}_{\theta_z}$,

$$\mathcal{R}_{\theta_z}=\exp(\theta_z:L_z:), \tag{48}$$

and second, by rotating a bend, we create a small component of the B field in the x direction of magnitude $\sin(\theta_z)/\rho_{12}$. The effect of this component is to leading order given by the map

$$\exp\left(-\frac{\sin(\theta_z)L_{12}}{\rho_{12}}y\right)\cong\exp(-\theta_z\Phi_{12}y). \tag{49}$$

We can then combine all of this into the following symmetrized expression for the rotated map:

$$\mathcal{R}_{\theta_z}[\mathcal{S}]\cong\mathcal{R}_{\theta_z}\exp\left(-\frac{\theta_z}{2}\Phi_{12}y\right)\mathcal{S}\exp\left(-\frac{\theta_z}{2}\Phi_{12}y\right)\mathcal{R}_{\theta_z}^{-1}. \tag{50}$$

This derivation seems to depend on the value of the bending field, which we related to the layout angle for an ideal magnet. However, as suggested by the final answer and demonstrated in the above sections of this paper, the small bending introduced by the tilt is a property of the layout alone. It is there for all possible maps $\mathcal{S}$ one may stick in the layout. Let us get the result of Eq. (50) using the locally exact representation. First one notices that the vector $\vec{D}$ given by Eq. (47) vanishes for an $x-y$ rotation. Thus we only have to examine $\vec{\beta}^{\text{in}}$ and $\vec{\beta}^{\text{out}}$:

$$\vec{\beta}^{\text{out}}=Y^{1/2}\vec{\beta}\approx\begin{pmatrix}1 & 0 & \frac{\Phi_{12}}{2}\\ 0 & 1 & 0\\ -\frac{\Phi_{12}}{2} & 0 & 1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\text{for a small layout angle}=\begin{pmatrix}\frac{\Phi_{12}}{2}\\0\\1\end{pmatrix},$$

$$\vec{\beta}^{\text{in}}=Y^{-1/2}\vec{\beta}\approx\begin{pmatrix}-\frac{\Phi_{12}}{2}\\0\\1\end{pmatrix}. \tag{51}$$

Thus, following the results of Eq. (42), we get for the rotated map

$$\mathcal{R}_{\theta_z}[\mathcal{S}] \cong \exp\left(\theta_z : L_z - \frac{\Phi_{12}}{2} L_x : \right) \mathcal{S} \exp\left(\theta_z : L_z - \frac{\Phi_{12}}{2} L_x : \right). \tag{52}$$

Using the dynamical expression of L_x given in Eq. (21), we obtain

$$\begin{aligned}
&\exp\left(\theta_z : L_z - \frac{\Phi_{12}}{2} L_x : \right) \\
&= \exp\left(\theta_z : L_z - \frac{\Phi_{12}}{2} y \sqrt{(1+\delta)^2 - p_x^2 - p_y^2} : \right) \\
&\cong \exp\left(\theta_z : L_z - \frac{\Phi_{12}}{2} y : \right) \cong \exp(\theta_z : L_z :) \exp\left(: - \frac{\theta_z \Phi_{12}}{2} y : \right) \\
&= \mathcal{R}_{\theta_z} \exp\left(: - \frac{\theta_z \Phi_{12}}{2} y : \right),
\end{aligned} \tag{53}$$

from which Eq. (50) immediately follows. Here we have made use of the smallness of the angles involved including the momenta.

The reader with a purely analytical mind may say to himself the following: ‘‘The author has derived formula (50) in two different ways, so what? Once we have the result we should just apply it independent of its source or origin.’’ Although this is true, *only* the correct formalism gives us a way to approximate the Euclidean group for small misalignments and tilts without breaking the ‘‘object orientedness’’ of a well-written code. Thus in a large machine with small bending angles it is preferable to keep the structure of Eq. (42) and to approximate the various operators used in this formula.

In conclusion, while it is nice to see usual results emerging from the correct formalism, this is not the main message of this paper. The primary message is for those who use object-oriented programming. One should use a theory in which the ‘‘magnet’’ is an object from a graphical as well as a dynamical point of view. And, when acceptable approximations are introduced in the code, they should be done in such a way as to preserve the magnet-object structure. One should ask questions like ‘‘Can I put back the more precise formulas easily?,’’ ‘‘Can I introduce the more complex small machine integrators without restructuring the program?,’’ or ‘‘Can I compute the effects of radiation and the stochastic beam envelopes without yet another rewriting?’’ It is a difficult task to keep all these questions in sight while designing an ideal code, however if at any moment one loses the ability to rotate a magnet independently of the nature of its single particle map, one can safely bet that the structure of the theory has been needlessly compromised in the code implementation.

ACKNOWLEDGMENTS

I am particularly grateful to Leo Michelotti for preaching the object oriented approach to accelerator physics. In addition, I am grateful to John Irwin, Yunhai Cai, and Chris Iselin for listening to my ideas. I recognize Alex Dragt’s original work concerning the use of dynamical patches (PROT as he called it) in accelerator simulations and for his suggestions in making an unreadable manuscript less unreadable. Finally I am immensely grateful to Stefania Petracca and, most of all, to Shuji Matsumoto of KEK for their line by line critique of the final manuscript.

[1] E. Forest, M. F. Reusch, D. Bruhwiler, and A. Amiry, Part. Accel. **45**, 66 (1994).